Name:____ Class: __________

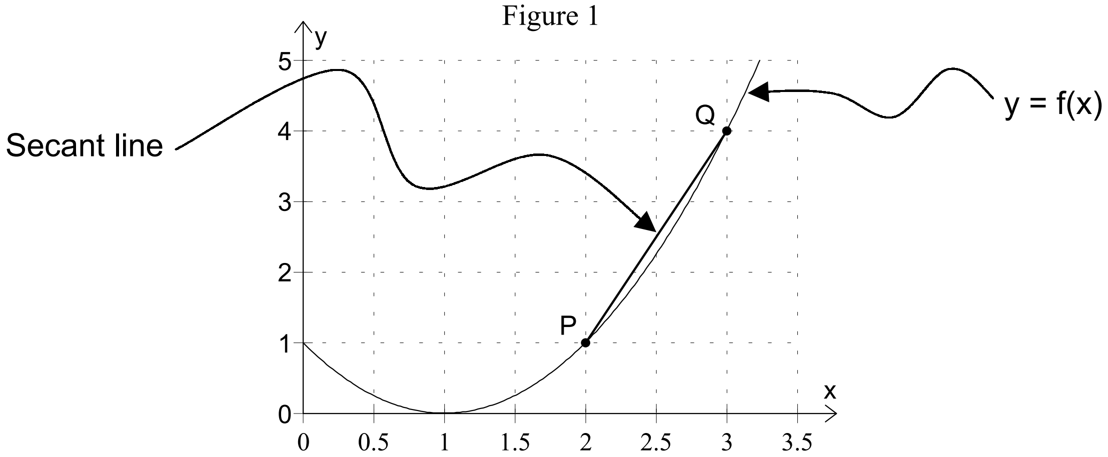
Calculus - Activity 1

Rate of change of a function at a point.

1) Start Maths Helper Plus, then load the file: 'Calculus - Derivatives 1.mhp'.

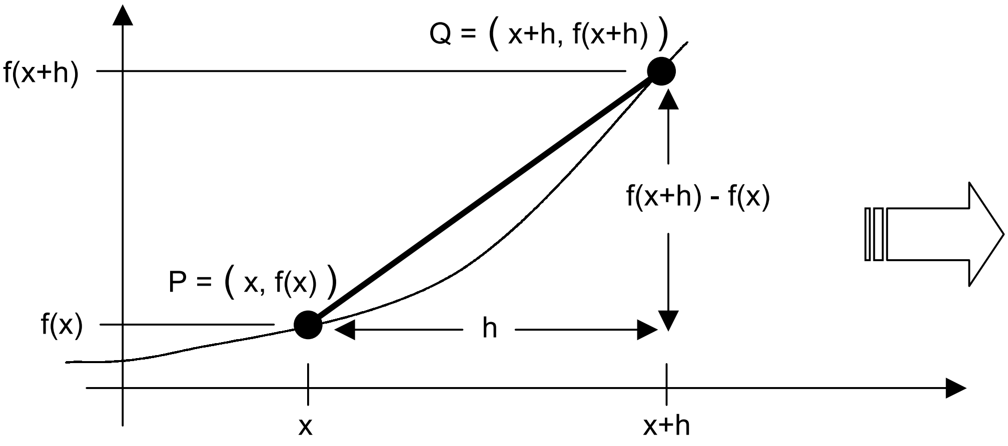
The graph view displays a graph of the function: $y = f(x) = x^2 - 2x + 1$ (See figure 1, below:)

Figure 1



We wish to find the gradient of the curved function line at 'P' where 'x' = 2. Another point 'Q' also lies on the curve. The two points define a straight 'secant' line. If Q is very close to P, then the gradient of the secant will be a good approximation of the gradient of the curve at 'P'.

The gradient of the secant can be found as follows:



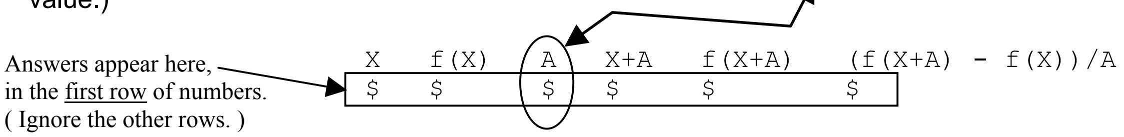
Thus the gradient of the secant is given by:

$$\frac{f(x+h) - f(x)}{h}$$

2) Complete this table. Use the 'P' and 'Q' points in figure 1 and the diagram and formula above.

x	f(x)	h	x+h	f(x+h)	f(x+h) - f(x)	gradient of secant

3) Use Maths Helper Plus to correct your answers. Hold down a Ctrl key while you press the 'T' key. The data set titled: 'Gradient of secant' displays the calculations. (Note that 'A' is used instead of 'h', and 'X' for the 'x' value.)



```
X   f(X)   A   X+A   f(X+A)    (f(X+A) - f(X))/A
$   $      $   $     $         $
```

Answers appear here,
in the first row of numbers.
(Ignore the other rows.)

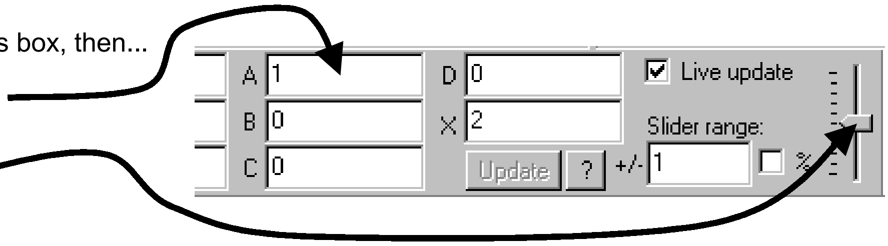
To see the graph again, use Ctrl 'G' for just the graph, or Ctrl 'B' for both text and graph together.

We want to estimate the gradient of the tangent line to the function y = f(x) = $x^2 - 2x + 1$ at point ‘P’ = (2,1). To do this, we will move point ‘Q’ along the curve towards ‘P’. This will make the secant PQ shorter so that its gradient will more closely match the gradient of the tangent line at ‘P’.

4) Press F5 to display the parameters box, then...

a) Click here to select variable ‘A’

b) Click here to select the slider



A 1
B 0
C 0
D 0
X 2
☑ Live update
Slider range:
+/- 1

With the slider selected, you can use the up and down arrow keys on the keyboard to change ‘A’.
If the arrow keys stop responding, click the slider button again.

5) Hold down ‘Ctrl’ and press ‘B’ to view both text and graph together. Use the mouse to adjust the splitter bar that divides text and graph views so that you can see both the text view calculations and the secant line on the graph.

6) Use the arrow keys to change ‘A’ as described in step 4, and so complete the table below.

h = ‘A’	gradient of secant: $\frac{f(x+h)-f(x)}{h}$ = `(f(X+A) - f(X))/A`
1	
0.5	
0.1	
0.01	
0.0001	
0	

To set these values:
- Click on the ‘A’ edit box
- Type the value for ‘A’
- Click the ‘Update’ button

7) Why can’t we calculate the gradient when h = 0 ?

8) As ‘h’ gets closer to zero, the gradient of the secant line approaches the actual gradient of the curve at point ‘P’.

From your results in the table above, estimate the gradient of the curve at ‘P’ = (2,1): ___________

The gradient of the curve at ‘P’ is called the ‘limit’ of $\frac{f(x+h)-f(x)}{h}$ as ‘h’ approaches zero.

This can be written like this: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, so in this case, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ = ___________

The gradient of a function at a point is called the derivative of the function with respect to the variable that approaches a limiting value. In this case, variable ‘x’ is approaching 2, so we have found the derivative of the function with respect to ‘x’ at x = 2.

For the function: y = f(x), the derivative of f(x) with respect to ‘x’ is written as: $\frac{dy}{dx}$

So for the function: y = $x^2 - 2x + 1$, we write: $\frac{dy}{dx}$ = ___________ at x = 2.

9) Set parameter ‘A’ to –1, then slowly increase ‘A’ towards zero.

Does `(f(X+A) - f(X))/A` approach the same limit as before ?

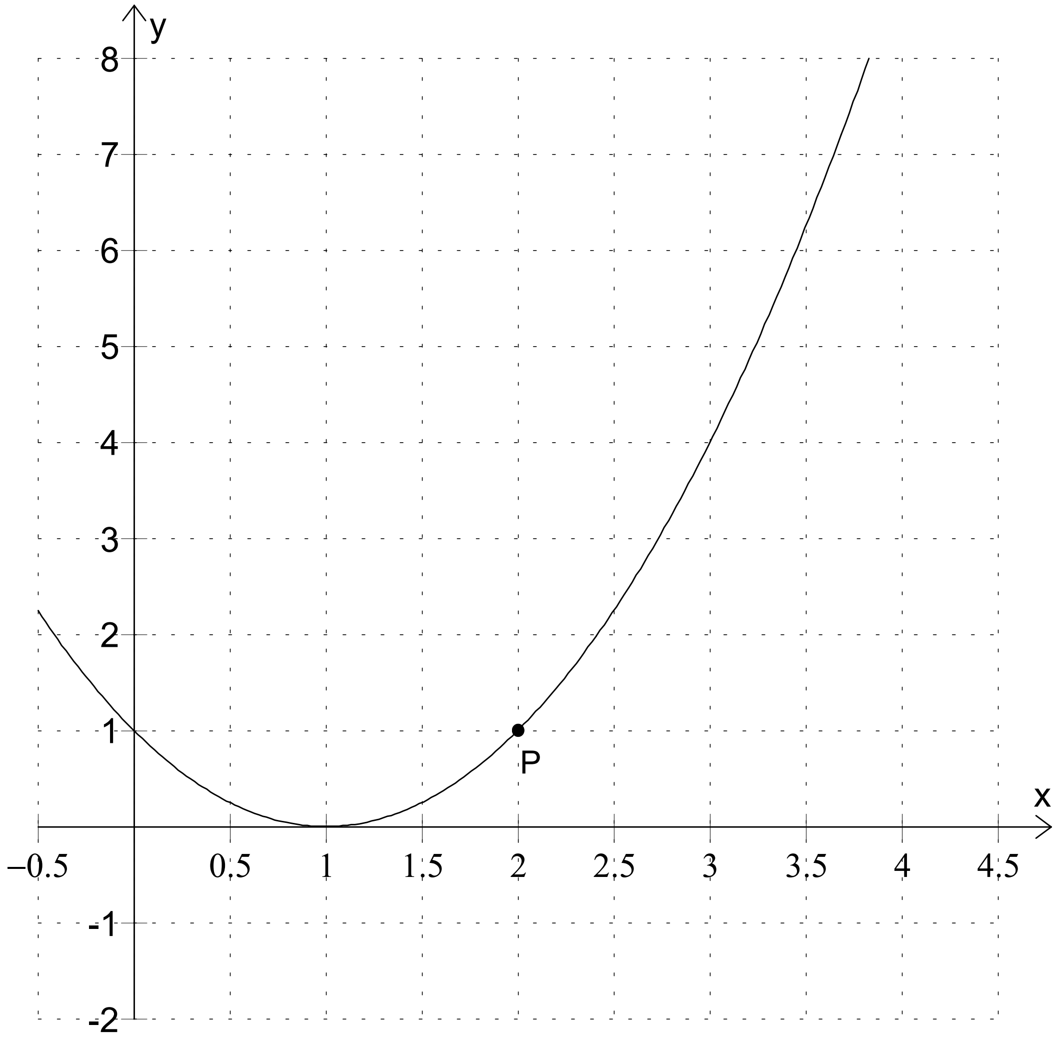
Name:____ Class: ___________

Calculus - Activity 2

Plotting the derived function.

1) Start Maths Helper Plus, then load the file: 'Calculus - Derivatives 2.mhp'.

The graph view displays a graph of the function: $y = f(x) = x^2 - 2x + 1$ (See below:)

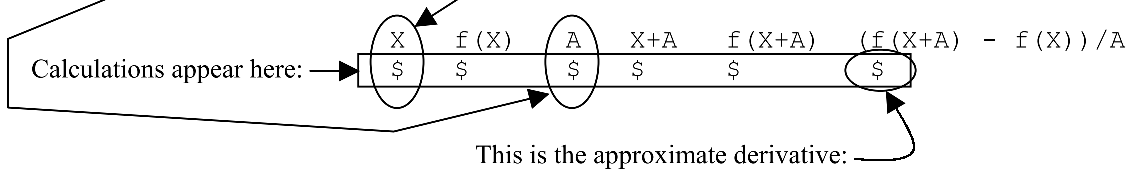


The derivative of a function y = f(x) with respect to 'x' is defined as: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The derivative also equals the gradient of the tangent to the function.

The text view displays the approximate derivative of the function near the point: (X, f(X)).
(NOTE: We are using <u>BIG 'X'</u> to represent a particular 'x' value.)

The data set titled: 'Gradient of secant' displays the calculations in the positions shown below:
Note that 'A' is used instead of 'h', and 'X' for the 'x' value.



2) From the approximate derivative shown on the text view, estimate the true derivative of $y = x^2 - 2x + 1$ at x = 2.

At x = 2, $\frac{dy}{dx}$ = ________________

NOTE: Sometimes, $\frac{dy}{dx}$ is written as: $f'(x)$, so we could also write: $f'(2)$ = ________________

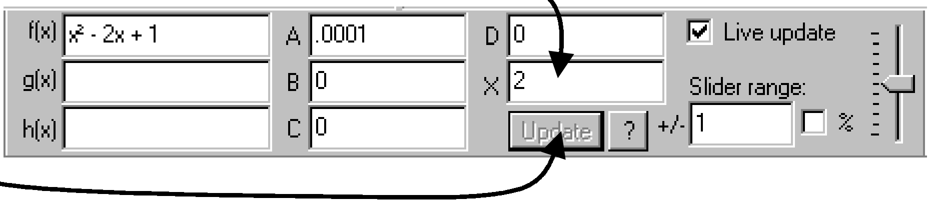
3) Plot the point: (2, f '(2)) on the graph on the other side of this sheet.

4) Press F5 to display the parameters box. (See below.)

To change the value of ‘X’, you...

a) Click on the ‘X’ edit box

b) Click the ‘Update’ button



f(x) $x^2 - 2x + 1$

g(x)

h(x)

A .0001

B 0

C 0

D 0

X 2

☑ Live update

Slider range: +/- 1 ☐ %

5) Change ‘X’ to each of the ‘x’ values in this table, then complete using the text view calculations:
(Round the values to two decimal places.)

Copy from question 2 →

x	f '(x)
0	
1	
2	
3	
4	

6) Plot the points: (x, f ‘(x)) from the table in question 5 on the graph paper on the front side of this sheet, then rule a line through the points.

7) What is the equation of the line through these points ? y = ______________________

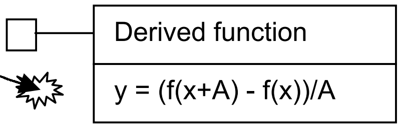
8) The function you obtained in question 7 is called the ‘derived function’ with respect to ‘x’, or the ‘derivative’ with respect to ‘x’.

The ‘y’ value of this function at a given ‘x’ value is the gradient of the function: $y = x^2 - 2x + 1$ at that ‘x’ value, so we normally write: $\frac{dy}{dx}$ = ______________________ or f '(x) = ______________________.

The process of finding the derived function is called differentiating.

9) Check your derivative plot with Maths Helper Plus.
The data set called ‘Derived function’ calculates the derivative for many points on along y = f(x). It then plots the points and joins them with lines to make a smooth curve.

- Double click on the text view beside the data set:
- Select the options tab: ‘Plot settings’
- Click on the ‘y’ check box
- Click OK



Derived function

y = (f(x+A) - f(x))/A

The derived function will now display as a red line. Compare with your graph.

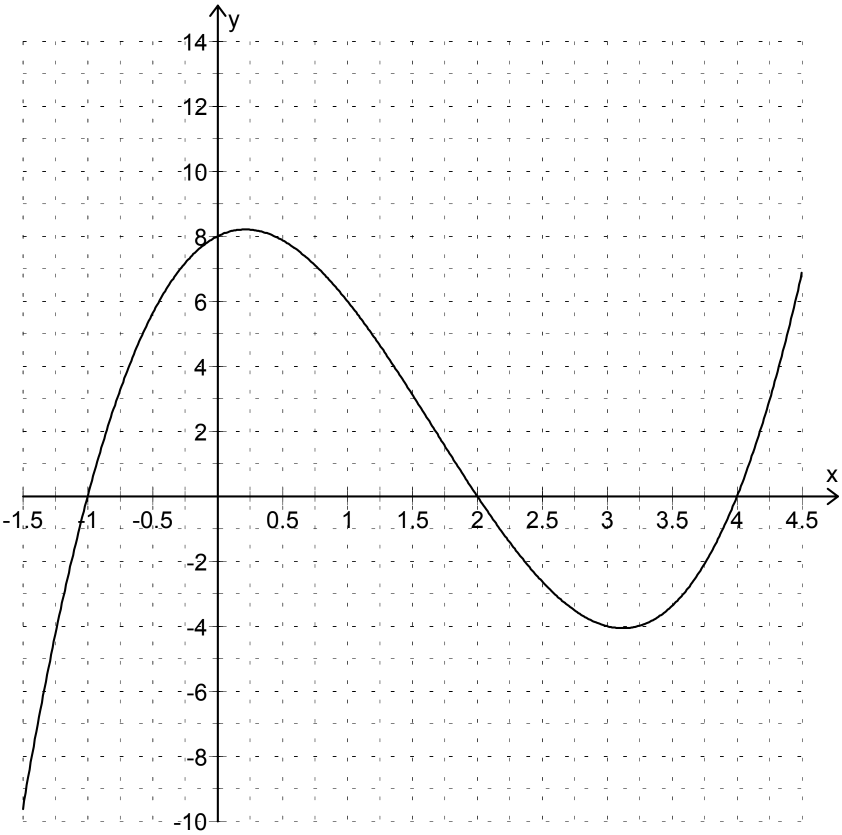
Name:____ Class: ___________

Calculus - Activity 3

Investigating the derived function by measuring gradients around a function curve.

1) Start Maths Helper Plus, then load the file: 'Calculus - Derivatives 3.mhp'.

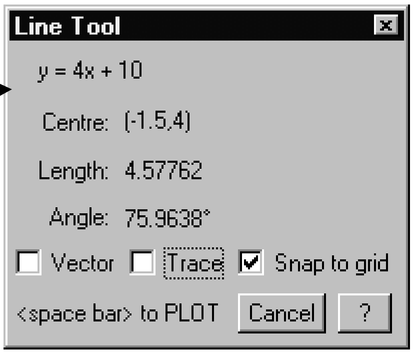
The graph view displays a graph of the function: $y = f(x) = x^3 - 5x^2 + 2x + 8$ (See below:)



At any given 'x' value, the derived function equals the gradient of the original function.
You will use the Maths Helper Plus 'line tool' to make quick estimates of the gradient of the plotted function at various points.

2) Select the line tool by clicking this toolbar button:

The line tool dialog box will be displayed, as shown:
If it covers any of the plotted graph, drag on the title bar at the top of the line tool dialog box to move it to a more convenient location.



Line Tool

$y = 4x + 10$

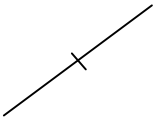
Centre: (-1.5,4)

Length: 4.57762

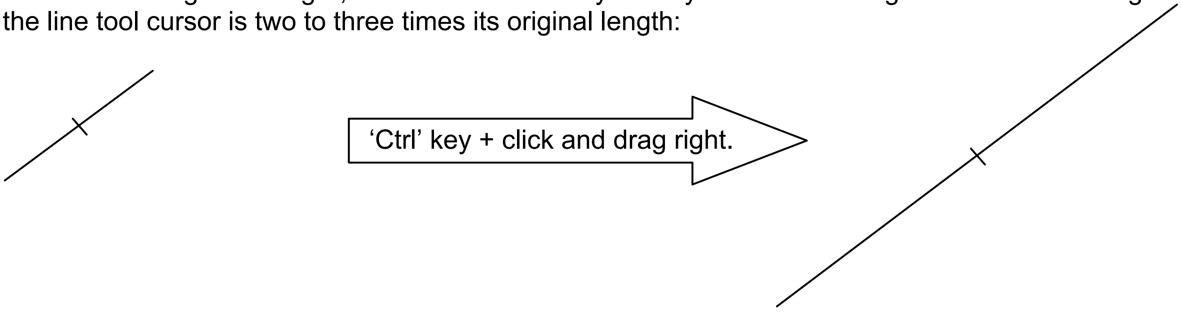
Angle: 75.9638°

☐ Vector ☐ Trace ☑ Snap to grid

3) Move the mouse cursor over the graph view. The line tool cursor will display on the graph view like this:



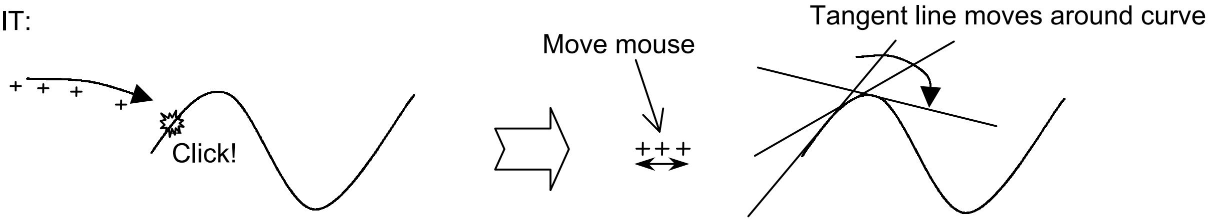
To make this line segment longer, hold down a 'Ctrl' key while you click and drag the mouse to the right. Do this until the line tool cursor is two to three times its original length:



4) Click to select the 'Trace' option on the line tool dialog box.

5) Click the '+' mouse cursor on any part of the plotted graph to find the approximate tangent at that point. A tangent line will lock to the function plot. Move the tangent along the function by moving the mouse left and right, or click on any other part of the function graph to immediately lock the tangent to that point.
HINT: Hold down the Shift key while tracing for finer control.

TRY IT:



6) The tangent line equation is displayed at the top of the line tool dialog box in the form: $y = mx + c$.
Use the line tool to measure the gradient ('m') of the function graph close to the given 'x' values. Write the gradients in the second column of the table below. Round the values to two decimal places.

x	gradient, 'm' = $\frac{dy}{dx}$	gradient sign (+ or –)	gradient change (increasing or decreasing)
-0.5			
0			
0.5			
1			
1.5			
2			
2.5			
3			
3.5			
4			

7) On the graph on the front of this sheet, plot the points (x, dy/dx) that you recorded in the table. Join the points with a smooth curve. This curve represents the derivative of $y = x^3 - 5x^2 + 2x + 8$ from $x=-0.5$ to $x=4$.

8) In the third column of the table, write '+' or '–' for positive or negative gradients.
A positive gradient means that as 'x' increases, the function goes uphill. Inspect the graph and verify that the derivative graph is positive when the function $y = x^3 - 5x^2 + 2x + 8$ is going uphill as 'x' increases.
What is happening to the original function curve when its derivative graph is:
(a) negative (b) zero

9) In the last column of the table, write 'Increasing' if that gradient is greater than the previous gradient (Above it in the table), or 'Decreasing' if it is less.
Use a coloured pencil or marker on the graph of $y = x^3 - 5x^2 + 2x + 8$ on the front of this sheet to indicate where the gradient is decreasing. Describe the behaviour of the derivative graph when the gradient of $y = x^3 - 5x^2 + 2x + 8$ is:
(a) decreasing (b) increasing

10) The point on $y = x^3 - 5x^2 + 2x + 8$ where the gradient changes from decreasing to increasing is called a point of inflection. How can you identify a point of inflection from the derivative graph ?

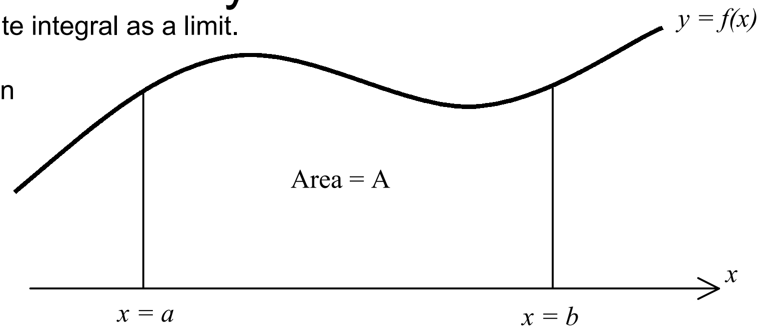
By inspection, estimate the approximate point of inflection on $y = x^3 - 5x^2 + 2x + 8$ and mark it with a small circle.

Name:______________________________________ Class: __________

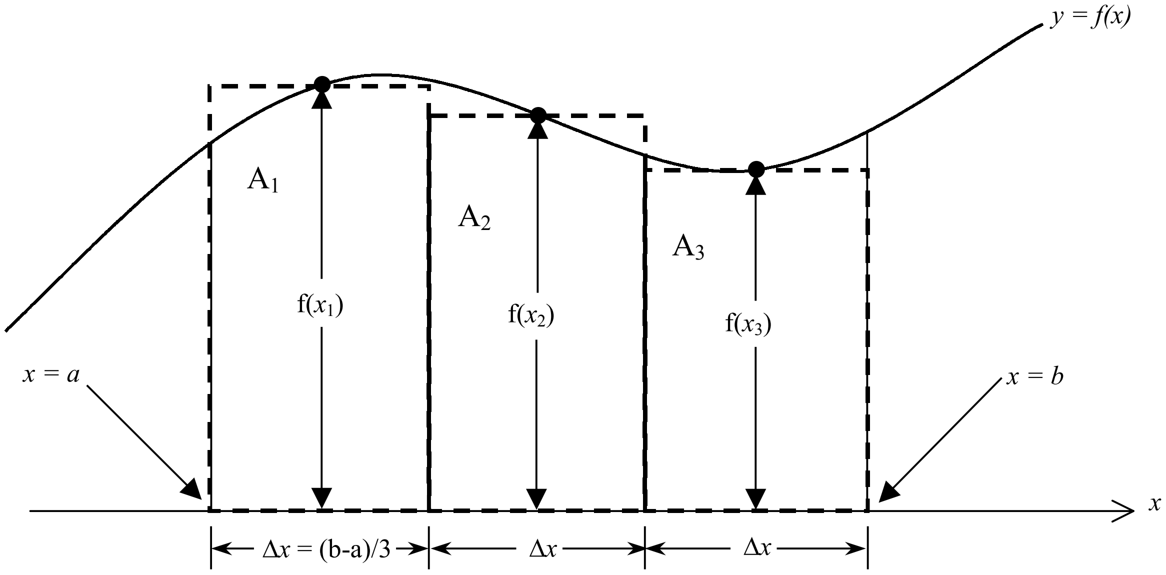
Calculus - Activity 4

The definite integral as a limit.

Consider the problem of finding the area 'A' between a curved line: y = f(x) and the 'x' axis, between x = a and x = b:



We can approximate the area by slicing it into rectangles and adding up the areas of the rectangles. In the diagram below, there are three rectangles of equal width. The height of each rectangle equals the height of the function graph at its midpoint, and the width of each rectangle = (b–a)/3:



The total area under the curve from x = a to x = b can be approximated by: $A = A_1 + A_2 + A_3$.

Notice that the area A1 is an overestimate of the area under the curve, A2 is about right, and A3 is too small. Thus this method may be accurate for some types of curves, but not for others. It depends on their shape.

One way to improve the accuracy for any shaped curve is to increase the number of rectangles. If there are 'n' rectangles, then as 'n' increases, the width of each rectangle decreases.

For very thin rectangles, areas at the top that don't fit the curve become insignificant, then the sum of the rectangles becomes very close to the area under the curved graph line.

Let F_a^b be the area under the graph: y = f(x) from x = a to x = b.

We can calculate the area by summing up the areas of the rectangular slices. If there are 'n' slices, then the total area is approximated by:

$$F_a^b \approx A_1 + A_2 + A_3 + \quad \cdots \quad + A_n = \sum_{i=1}^{n} A_i$$

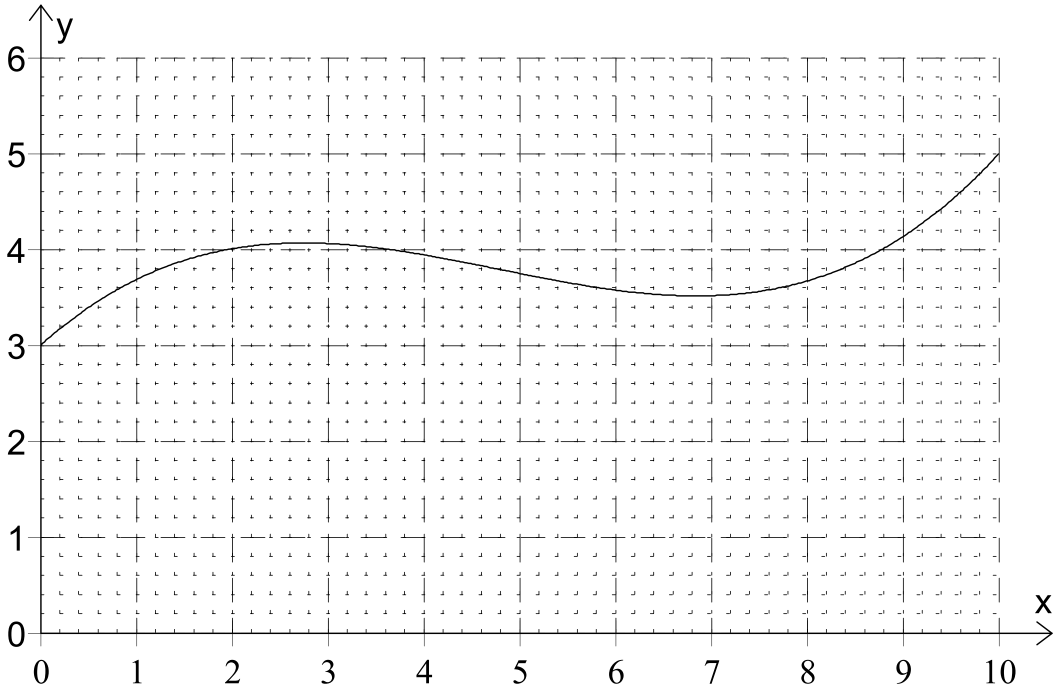
The true area can be written as the limit of this sum as 'n' approaches infinity, like this:

$$F_a^b = \lim_{n \to \infty} \sum_{i=1}^{n} A_i$$

We will now use Maths Helper Plus to calculate the area under a curved function graph by adding rectangles.

1) Start Maths Helper Plus and load the file: 'Calculus - Integrals 1.mhp'

The graph view will display the curved function line as shown below:



You will use the graph above to estimate the area under the graph from x = 1 to x = 9 using 4 rectangles.

2) Draw the rectangles on the graph above with the top middle point of each rectangle on the graph line, as shown on the front page.

3) What is the width, 'w', of each rectangle ? ___________

4) Calculate the area of each rectangle, then add the areas:

A1 = _______ A2 = _______ A3 = _______ A4 = _______ Total area = _______

5) Do you think your total is greater or less than the true area under the curve ?

You will now find a better estimate of the area under this curve by using much larger numbers of rectangles. Maths Helper Plus can do the calculations for you.

6) Move the tip of the mouse pointer onto the graph curve, then double click to display the options dialog. Select the 'Integrals' tab. Now set these options:

- In the 'Limits of integration' edit box, type: [1,9] to specify x = 1 to x = 9.
- In the 'Number of intervals' edit box, type several values of 'n', like this: 3,4,10,20,50,100,200,300
- Select: 'Shade areas on graph'.
- In 'Methods of integration', select: 'Rectangular (mid-point)'
- Set 'Calculation mode' to 'area'.
- Click OK to close the dialog box.

The text view will display the summed areas for the different values of 'n'.

7) Compare your answer for question 4 with the area calculated by Maths Helper Plus for n = 4.

8) What is the area under this function graph between x = 1 and x = 9 correct to four decimal places ? _________

9) About what value of 'n' is required to achieve this accuracy ? _________

Calculus - Activity 5

Primitives of polynomial functions by direct measurement of areas.

The 'fundamental theorem of the calculus' states that: $\int_a^b f(x)dx = F(b) - F(a)$

where '*F(x)*' is the primitive of the function '*f(x)*'.

'a' and 'b' are two 'x' values. *F(b) – F(a)*, called the 'definite integral', is a number related to the area between the curve and the 'x' axis, from x=a to x=b. Any part of this area above the 'x' axis is positive, while parts of the area below the 'x' axis are negative. Adding these 'positive areas' and 'negative areas' gives the definite integral. Thus the definite integral can be zero if equal amounts of area are above and below the 'x' axis between 'a' and 'b'.

Maths Helper Plus can easily measure the definite integral of a plotted function. You can use this to find the primitive function: *F(x)*.

The method is to choose some fixed 'x' value for 'a', then choose different values for x = 'b'.

Since any 'x' values can be chosen for 'b', we will replace 'b' with 'x':

$$\int_a^x f(x)dx = F(x) - F(a)$$

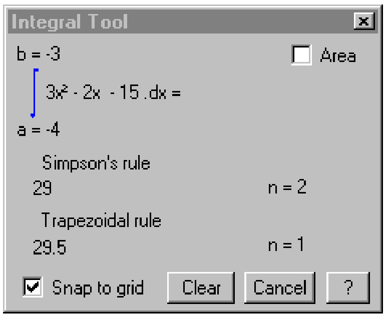
Adding F(a) to both sides we have:

$$F(x) = \int_a^x f(x)dx + F(a)$$

1) Start Maths Helper Plus, then load the file: 'Calculus - Integrals 2.mhp'.
The graph view will display a plot of the function: $y = 3x^2 - 2x - 15$ (See below.)

2) Select the 'Integral' tool by clicking this button: ∫ on the toolbar.

The integral tool dialog box will be displayed:



b = -3

☐ Area

$\int$ 3x² - 2x - 15 .dx =

a = -4

Simpson's rule

29 n = 2

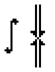
Trapezoidal rule

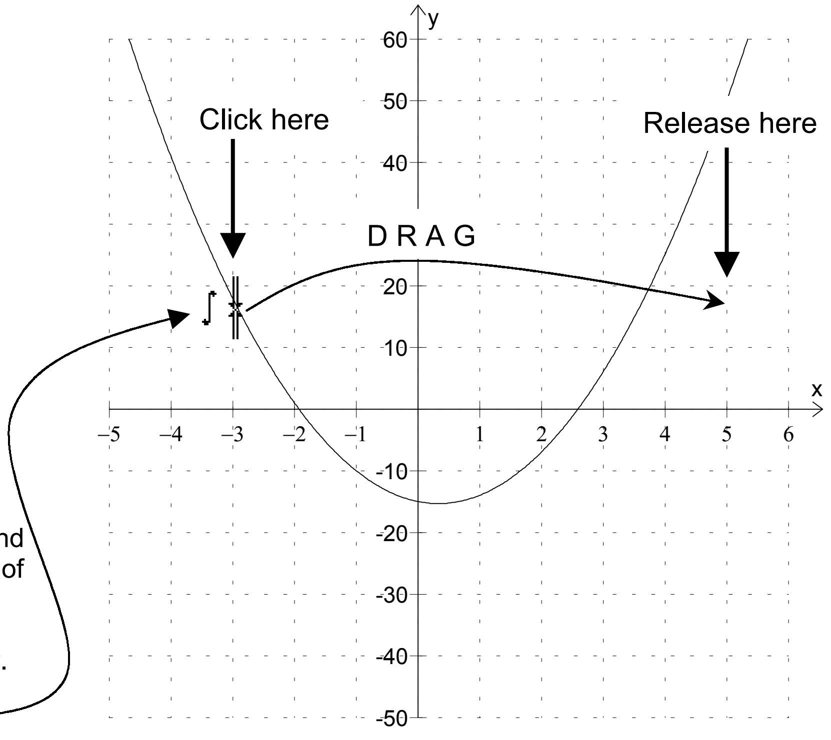
29.5 n = 1

☑ Snap to grid

Make sure the 'Area' option is NOT selected and 'Snap to grid' IS selected. Drag on the title bar of this dialog to move it to a convenient location.

3) Move the mouse cursor over the graph view. The integral tool cursor will display:





4) Practice finding a definite integral from a = -3 to b = 5. To do this, move the integral tool cursor so that the graph curve is between the two arrows and so that the arrows line up with the starting 'a' value of -3. (See above.)

Now click the and drag the mouse to the right until the graph is shaded all the way to 'x' = 5, then release the mouse button. Read the '<u>Simpson's rule</u>' value for the definite integral from the dialog box. (It should be 16)

5) Now to record some values of the definite integral: $\int_a^x f(x)dx$

Let ‘a’ be -3, and ‘x’ be integers from -3 to 5.

(Click the ‘Clear’ button on the integral tool dialog box to clear the last measurement ready to start another.)

To find a definite integral, click on the curve at a = -3 as in (4) above, and drag the mouse right until you reach the required ‘x’ value.

Release the mouse and record the definite integral value in the table.

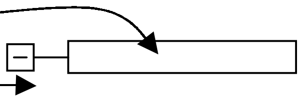
When you have finished recording values, click the ‘Cancel’ button on the integral tool dialog box.

x	definite integral from -3 to ‘x’
-3	0
-2	
-1	
0	
1	
2	
3	
4	
5	16

6) Plot your definite integral values in the table as a set of (x,y) points, with the integrals as the ‘y’ coordinates. For example, the integral from x = -3 to x = 5 is 16, so this would be the point (5,16).

To plot points you:

- Click on the input box. (On the text view.)
- Type the points to plot, like this: (-3,0) (-2, ...) ...
- Click outside of the input box.



7) With the mouse pointer near the exact centre of one of the plotted points on the graph view, double click the mouse to display the options dialog for the points. Select the ‘Curve Fitting’ tab and select the ‘cubic’ option. Click the ‘Plot’ button, then ‘OK’ to close the dialog box.

8) Copy the ‘Best fit cubic equation’ from the text view:

y = ____

9) Round off all numbers in this equation to whole numbers or whole number fractions.

y = ________________________

10) All primitive functions include an unknown constant term, often written as ‘c’.

In our definition: $F(x) = \int_a^x f(x)dx + F(a)$, the unknown ‘F(a)’ term is also a constant value. (Why ?)

Therefore, simply replace the constant term in (9) above with letter ‘c’ to represent any constant.
This is then the primitive function:

F(x) = ______________________

Extension Activity

11) Repeat 5 to 10 above, but with ‘a’ = -2. Measure the definite integral from -2 to ‘x’ = -1, 0, 1, 2, 3, 4, 5.

12) Do you end up with the same primitive function: F(x) ? ________

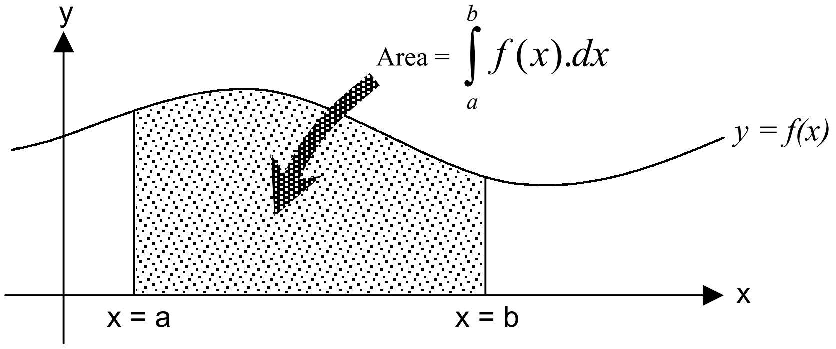
Why is this ?

Name:______________________________________ Class:__________

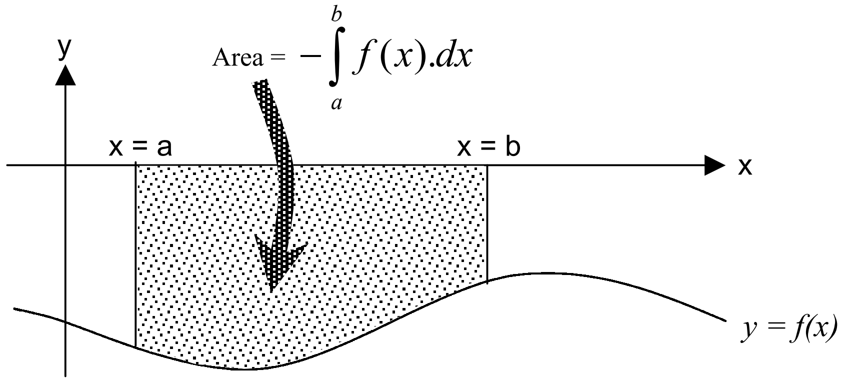
Calculus - Activity 6

Area under a curve.

The definite integral can be used to find the area between a graph curve and the 'x' axis, between two given 'x' values. This area is called the 'area under the curve' regardless of whether it is above or below the 'x' axis. When the curve is above the 'x' axis, the area is the same as the definite integral ...

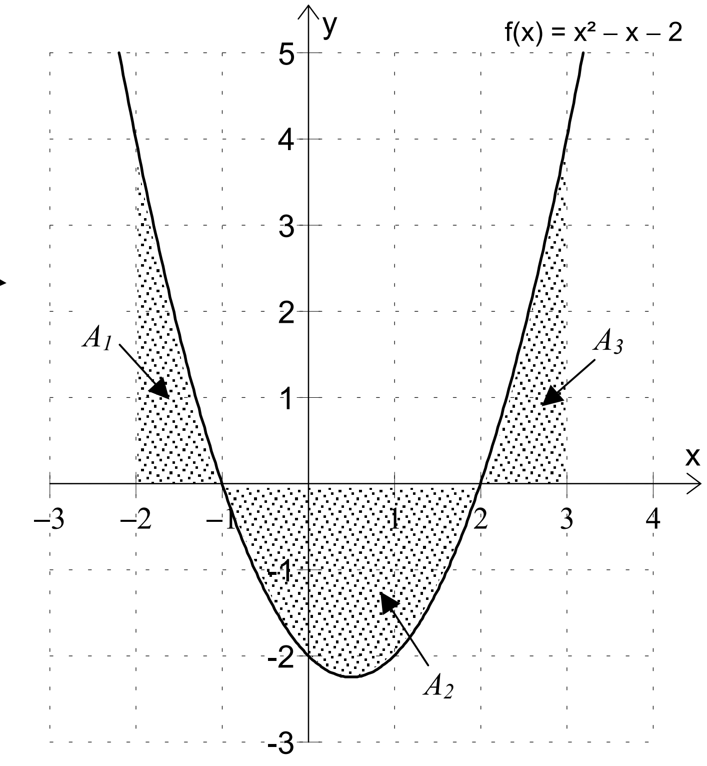


but when the graph line is below the 'x' axis, the definite integral is negative. The area is then given by:



Sometimes part of the graph is above the 'x' axis and part is below, then it is necessary to calculate several integrals. When the area of each part is found, the total area can be found by adding the parts.

For example, to find the area between the graph of: $y = x^2 - x - 2$ and the 'x' axis, from $x = -2$ to $x = 3$, we need to calculate three separate integrals: →



The zeros of the function f(x) that lie between –2 and 3 form the boundaries of the separate area segments. In this case there are zeros at $x = -1$ and $x = 2$, and so three separate areas must be found: A_1, A_2 and A_3 as follows:

$$A_1 = \int_{-2}^{-1} \left(x^2 - x - 2\right) dx$$

$$A_2 = -\int_{-1}^{2} \left(x^2 - x - 2\right) dx$$

$$A_3 = \int_{2}^{3} \left(x^2 - x - 2\right) dx$$

→ So the shaded area, $A = A_1 + A_2 + A_3$.

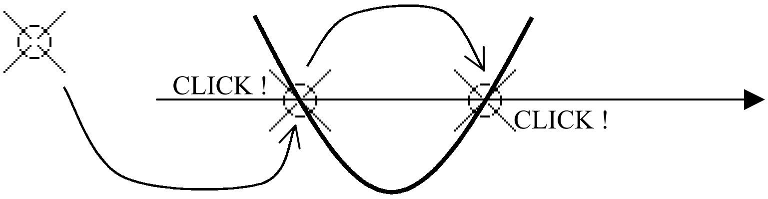
Do NOT use the integral tool for this activity!

Maths Helper Plus can graph the function, locate the zeros and calculate the definite integrals. Follow these steps to find the area under the curve as described in the example on the front side of this sheet.

1) Start Maths Helper Plus, then load the file: ‘Calculus - Integrals 3.mhp’.
The graph view will display a plot of the function: $y = x^2 - x - 2$.

2) We want to find the area under the curve between $x = -2$ and $x = 3$. The curve crosses the ‘x’ axis between these ‘x’ values, so we use the ‘intersection tool’ to locate the zeros, like this:

Click the toolbar button to select the intersection tool. Now click the mouse cursor on the points where the graph cuts the ‘x’ axis.
In each case, read and record the ‘x’ coordinate of the intersection point from the dialog box.



Zeros at: x = ________ and at x = ________

Cancel the intersection tool by clicking the ‘Cancel’ button in the dialog box.

3) Identify the separate integrals that need to be found. (There are three in this case.) Write down the boundary ‘x’ values for each of these integrals:

Integral 1: from ________ to ________.

Integral 2: from ________ to ________.

Integral 3: from ________ to ________.

4) Carefully point to the function curve with the mouse pointer then double click with the mouse to display the options dialog for the function. (This works better on not-so-steep parts of the graph.)

Select the ‘Integrals’ tab, then click on the ‘Limits of integration’ edit box. Type the integral boundaries you require using square brackets for each separate integral, like this: [-2,-1] [-1,2] [2,3]

Click on the ‘Number of intervals’ edit box. Type : 100. (The larger this number the greater the accuracy, but if it is too big the calculations may be slow. This must be an even number to use Simpson’s rule.)

Select ‘Shade areas on graph’. Also select ‘Simpson’s rule’. ‘Calculation mode’ should be ‘definite integral’.

Click the ‘OK’ button to close the dialog box. The areas will be shaded on the graph, and the integrals will be displayed on the text view.

5) Add the absolute value of the definite integrals to calculate the total required area.

Total area = ____________ + _____________ + _____________ = _____________

6) Use Maths Helper Plus to find the area under the functions below in the interval shown. Use this table to record your results. (The first row has been partly filled in as an example.)

function	interval	zeros at x =	integral boundaries	calculated areas	total area
$y = x^2 - x - 2$	–2 to 3	–1, 2	[–2,–1] [–1,2] [2,3]		
$y = -x^2 + 2x + 3$	–2 to 2				
$y = x^2 - 4$	–3 to 3				
$y = x^3 + x^2 - 2x$	–2.5 to 1.5				

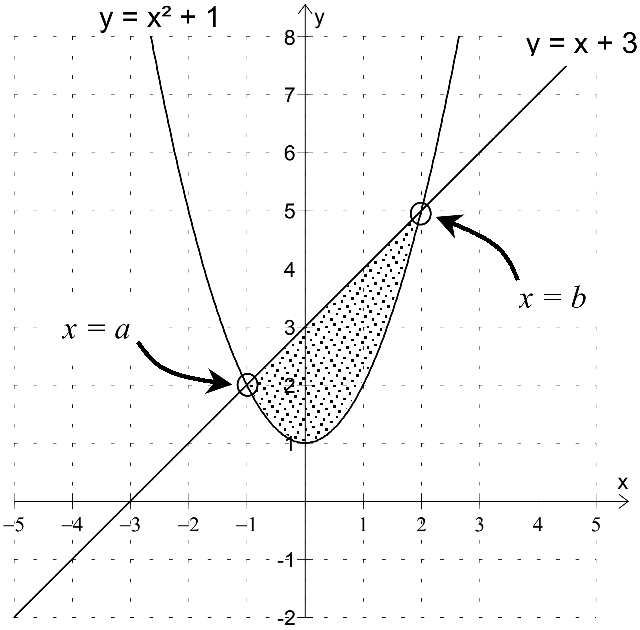
Calculus - Activity 7

Area between two curves with two intersection points by subtraction of areas.

To find the area between two intersecting curves that only intersect at two points, we first find the 'x' coordinates of the two intersection points: $x = a$ and $x = b$. Definite integrals give us the area under each curve from $x = a$ to b, then we subtract the two areas.

1) Start Maths Helper Plus, then load the file:
'Calculus - Integrals 4.mhp'.

The graph view will display a simultaneous plot of the two functions: $y = x^2 + 1$ and $y = x + 3$ ⟶



2) Click the ☒ toolbar button to select the intersection tool. Click the mouse cursor at the intersection points of the two curves.

Record the 'x' coordinates of the intersection points:

x (left) = 'a' = ______________

x (right) = 'b' = ______________

Click 'Cancel' on the intersection tool dialog to cancel the intersection tool.

4) Carefully point to one of the function curves with the mouse pointer. Double click to display the options dialog box for that function. (This may work better on parts of the graph that are not so steep.)

- Select the 'Integrals' tab, then click on the 'Limits of integration' edit box. Type the 'a' and 'b' values from (2) above with square brackets, like this: [-1,2]
- Click on the 'Number of intervals' edit box. Type : 100. (The larger this number the greater the accuracy, but if it is too big the calculations may be slow. This must be an even number to use Simpson's rule.)
- Select 'Shade areas on graph'. Also select 'Simpson's rule'. 'Calculation mode' should be 'definite integral'. Click the 'OK' button to close the dialog box.

The area under this function will be shaded on the graph, and the definite integral will be displayed on the text view.

The absolute (positive) value of the definite integral equals the area of the first function, = ________________

5) Carefully point to the other function curve and double click as before.

- Select the 'Integrals' tab and set the same options as for the first function.
- Click the 'Shade colour...' button, and choose a very pale colour that is different to the shading colour of the other function. Pale blue works well. Click the 'OK' button to close the dialog box.

The absolute value of the definite integral equals the area of the second function, = ________________

6) The area between the two graphs will now have its own colour and be easy to identify.
Calculate the area between the curves by subtracting the smaller area from the larger area.

Area between curves = __________ – ___________

= __________

7) For each of the pairs of functions below, use Maths Helper Plus to help you find the area between the plotted curves. In each case, write answers and calculations in the spaces provided. Also, sketch the graphs on the graph area provided, and shade the area between the curves.

In each case, start by creating a new document in Maths Helper Plus. (‘new’ command in the ‘File’ menu.)

a) Find the area between: $y = -x^2 + 2x + 3$ and $y = -1x + 3$

Intersection points: x (left) = ‘a’ = ____________ .

x (right) = ‘b’ = ______________ .

First function:

Definite integral from x = a to x = b = ______________.

Area under curve from x = a to x = b = ______________.

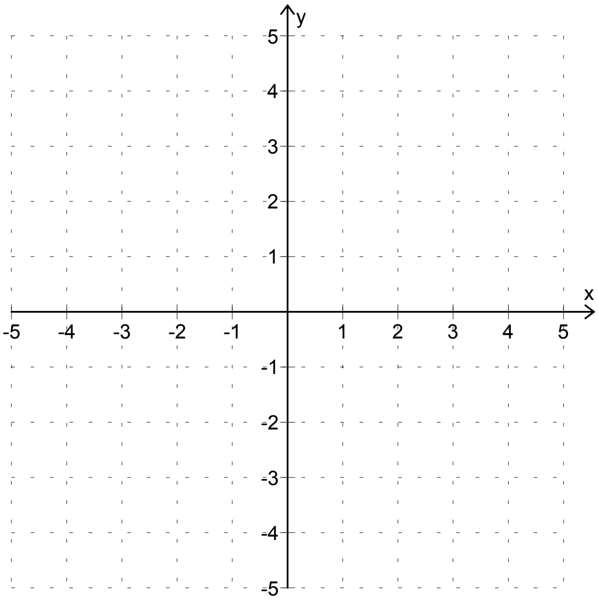
Second function:

Definite integral from x = a to x = b = ______________.

Area under curve from x = a to x = b = ______________.

Area between curves = bigger area – smaller area

= ________________.



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b) Find the area between: $y = x^2 - 1$ and $y = -x^2 + 2x + 3$

Intersection points: x (left) = ‘a’ = ____________ .

x (right) = ‘b’ = ______________ .

First function:

Definite integral from x = a to x = b = ______________.

Area under curve from x = a to x = b = ______________.

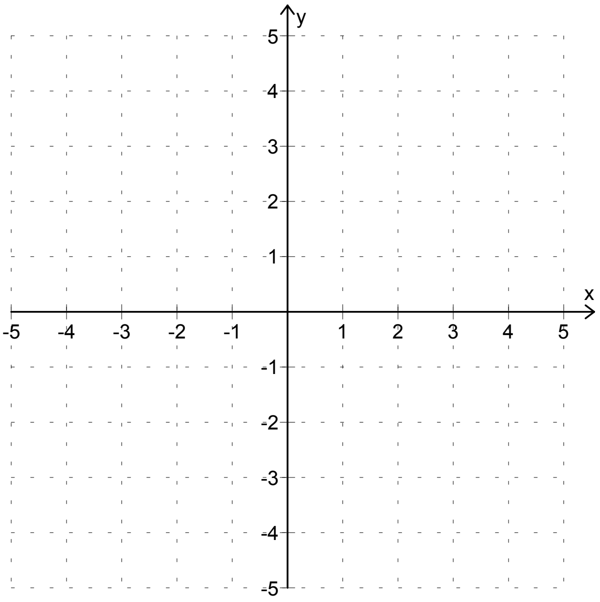
Second function:

Definite integral from x = a to x = b = ______________.

Area under curve from x = a to x = b = ______________.

Area between curves = bigger area – smaller area

= ________________.



CHALLENGE QUESTION

c) Find the area between: $y = x^2(x-2)(x-3)$ and $y = 0.5x + 0.5$